Light scattering by adjacent red blood cells - a mathematical model

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ABSTRACT

Simple approximate scattering theories such as the Rayleigh-Gans theory are not generally applicable to the case of light scattering by red blood cell (RBC) aggregates, including thrombus. This is mainly due to the extremely short distance separating erythrocytes in the aggregates (of the order of 25 nm) as well as to the substantial size of the aggregates. Therefore in this paper a new mathematical model predicting the electromagnetic field produced by the scattering of a plane electromagnetic wave by a system of two adjacent RBCs is presented. Each RBC is modelled as a homogeneous dielectric ellipsoid of complex index of refraction surrounded by transparent plasma. The relative position and orientation of the ellipsoids are arbitrary. Scattering is formulated in terms of an integral equation which however contains two singular kernels. The singular equation is transformed into a pair of nonsingular integral equations for the Fourier transform of the internal field of each RBC. The latter equations are solved by reducing them by quadrature into a matrix equation. The resulting solutions are used to estimate the scattering amplitude. Convergence aspects concerning the numerical calculation of the matrix elements originating from the interaction between the RBCs are also presented.

1. INTRODUCTION

Up to now most theoretical models simulating light scattering by red blood cells have been based on simple approximate theories[1] such as the Rayleigh-Gans theory[2]. Nevertheless simple theories are not generally applicable to the case of RBC aggregates (including thrombus) in the optical frequencies. This is mainly due to the extremely short distance separating erythrocytes in the aggregates (of the order of 25nm) as well as to the substantial size of the aggregates[3]. The dielectric properties of red blood cells and plasma[4] play an equally important role.

Therefore a rather complete theoretical analysis would be useful in order to study the effect of "high packing density" of the RBCs constituting the aggregates on the scattering of light. The current mathematical model deals with the case of two adjacent RBCs and is based on the general Fredholm integral theory[5,6] which can be universally applied.

Some of the areas where the model could be applied are the following:
Laser angioplasty. Estimation of the effect of thrombus on the laser light scattering during pertaining therapeutic processes.[7,8] **Laser RBC aggregometry**. Rheological characterization of blood by ellipsoidal deformation of the RBCs and laser diffraction measurements for diagnostic purposes.[9] **Flow cytometry.** Diagnostic quantification of various cell characteristics by measuring the intensity of light scattered by cells into specific angular intervals.[10]
Furthermore the suggested model is quite general so as to be applicable to non biomedical areas as well such as to astrophysics and atmospherical physics (e.g. light scattering by interstellar dust and aerosols.)

2. THE MODEL

Both RBCs are modelled as homogeneous dielectric ellipsoids of semiaxes a_1, b_1, c_1 and a_2, b_2, c_2 respectively. For $a_1 = b_1$, $c < a_1$ and $a_2 = b_2$, $c_2 < a_2$ the shapes of oblate spheroids which are closer to the shapes of normal RBCs are obtained. However the shape of a general ellipsoid seems to be a very good choice for the modelling of a sickle cell or even for the modelling of an entire rouleau of RBCs[3]. The relative (to plasma) index of refraction n_0 and the relative (to plasma) dielectric constant $\varepsilon = n_0^2$ of both RBCs for a given wavelength are considered complex. The index of refraction (and consequently the dielectric

 0-8194-1659-2/95/$6.00

constant) of the surrounding plasma are considered real, given that plasma has a very low optical absorption in the visible and near infrared regions of the spectrum.[4]

Dyadic notation is used throughout the analysis. The following three orthogonal coordinate systems are used: the absolute system xyz and the local coordinate systems $x_1y_1z_1$ and $x_2y_2z_2$ with their origins at the centres of the ellipsoids V_1 and V_2 respectively (see Fig 1).

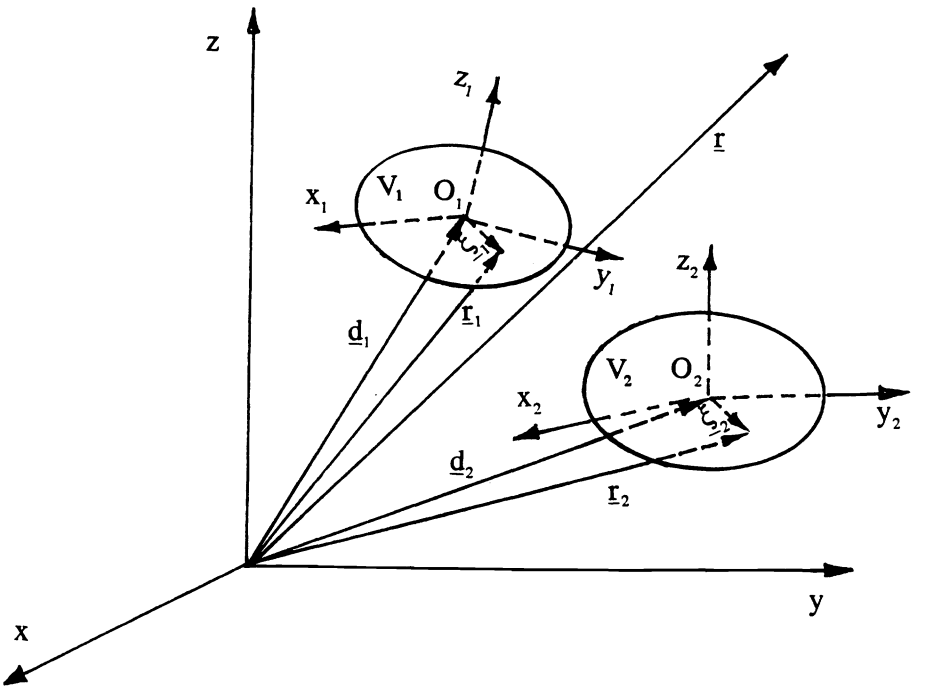

Fig. 1 The three coordinate systems used in the analysis

The ellipsoids V_1 and V_2 are defined in their corresponding local coordinate systems by the equations

$$\frac{x_1^2}{a_1^2}+\frac{y_1^2}{b_1^2}+\frac{z_1^2}{c_1^2}=1 \tag{1}$$

$$\frac{x_2^2}{a_2^2}+\frac{y_2^2}{b_2^2}+\frac{z_2^2}{c_2^2}=1 \tag{2}$$

The following symbol conventions are used throughout the analysis:

X denotes a scalar quantity (generally a comlex number)

$\underline{X}, \underline{\underline{X}}$ denote vectors (of generally complex elements)

$\hat{X}, \hat{\underline{X}}, \hat{\underline{\underline{X}}}$ denote unit vectors (of real elements)

$\overline{\underline{X}}$ denotes a dyadic (of generally complex elements)

A plane electromagnetic wave of wave vector $\underline{k}_i = k_0 \hat{k}_i$ and polarization $\hat{e}_i$ is incident on the system of the two ellipsoids. The time dependence is taken as exp(-iωt). The electric field (dyadic) at any point $\underline{r}$ of the surrounding medium (plasma) is given in the xyz coordinate system by [6]

$$\bar{\underline{E}}(\underline{r}) = \bar{\underline{J}}_i \exp(i\underline{k}_i \underline{r}) + \int_{V_1} d\underline{r}_1 \gamma \bar{\underline{G}}(\underline{r}, \underline{r}_1) \bar{\underline{E}}(\underline{r}_1) + \int_{V_2} d\underline{r}_2 \gamma \bar{\underline{G}}(\underline{r}, \underline{r}_2) \bar{\underline{E}}(\underline{r}_2) \tag{3}$$

where

$$\gamma = \frac{k_0^2}{4\pi}(\varepsilon - 1) \tag{4}$$

k_0 is the plasma propagation constant,

$$\bar{\underline{J}}_\lambda = \bar{\underline{I}} - \hat{k}_\lambda \hat{k}_\lambda \tag{5}$$

$$\bar{\underline{G}}(\underline{r}, \underline{r}') = (\bar{\underline{I}} + k_0^{-2} \underline{\nabla\nabla}) G(\underline{r}, \underline{r}') \tag{6}$$

$$G(\underline{r}, \underline{r}') = \frac{\exp(ik_0 |\underline{r} - \underline{r}'|)}{|\underline{r} - \underline{r}'|} \tag{7}$$

In order to obtain the plane plus scattered wave for an incident wave $\hat{e}_i E_0 \exp(i\underline{k}_i \underline{r})$, both sides of (3) should be multiplied by $\hat{e}_i E_0$. [11]

The $\underline{\xi}_1$ and $\underline{\xi}_2$ vectors (see Fig.1) are expressed in the local coordinate systems $x_1y_1z_1$ and $x_2y_2z_2$ respectively. The following relations hold

$$\underline{r}_1 = \underline{d}_1 + \bar{\underline{A}}_1 \underline{\xi}_1 \tag{8}$$

$$d\underline{r}_1 = d\underline{\xi}_1 \tag{9}$$

$$\underline{r}_2 = \underline{d}_2 + \bar{\underline{A}}_2 \underline{\xi}_2 \tag{10}$$

$$d\underline{r}_2 = d\underline{\xi}_2 \tag{11}$$

It can be considered that the xyz coordinate system originates from either the $x_1y_1z_1$ or the $x_2y_2z_2$ local system by rotation and translation.. In that case the $x_1y_1z_1$ system is rotated about its original z_1 axis by the ψ_1 angle ($0 \le \psi_1 \le 2\pi$), the transformed system is rotated about its new y_1 axis by the θ_1 angle ($-\pi/2 \le \theta_1 \le \pi/2$) and the latter system is rotated about its new x_1 axis by the ω_1 angle ($0 \le \omega \le 2\pi$). The rotated $x_1'y'z_1'$ system is translated by the vector $-\underline{d}_1$ so that the xyz coordinate system is obtained. In a similar manner the absolute xyz system can be obtained from the local $x_2y_2z_2$ coordinate system (Euler angles ψ_2, θ_2 and ω_2, translation vector $-\underline{d}_2$). Then the $\bar{\underline{A}}_1$ matrix is given by the relation

$$\underline{\overline{A}}_1 = \begin{bmatrix} \cos(\psi_1)\cos(\theta_1) & \sin(\psi_1)\cos(\theta_1) & -\sin(\theta_1) \\ (-\sin(\psi_1)\cos(\omega_1)+ & (\cos(\psi_1)\cos(\omega_1)+ & \\ +\cos(\psi_1)\sin(\theta_1)\sin(\omega_1)) & +\sin(\psi_1)\sin(\theta_1)\sin(\omega_1) & \cos(\theta_1)\sin(\omega_1) \\ (\sin(\psi_1)\sin(\omega_1)+ & (-\cos(\psi_1)\sin(\omega_1)+ & \\ +\cos(\psi_1)\sin(\theta_1)\cos(\omega_1)) & +\sin(\psi_1)\sin(\theta_1)\cos(\omega_1) & \cos(\theta_1)\cos(\omega_1) \end{bmatrix} \tag{12}$$

The $\underline{\overline{A}}_2$ matrix is given by equation (12) if all "1" subscripts are substituted by "2" subscripts.

The dyadic scattering amplitude $\underline{\overline{f}}(\underline{k}_s, \underline{k}_i)$ for scattering in the direction of $\underline{k}_s = k_0 \hat{\underline{k}}_s$ is defined by

$$\underline{\overline{E}}(\underline{r}) \xrightarrow[r\to\infty]{} \underline{\overline{J}}_i \exp(i\underline{k}_i \underline{r}) + \frac{\exp(ik_0 r)}{r} \underline{\overline{f}}(\underline{k}_s, \underline{k}_i) + 0(\frac{1}{r^2}) \tag{13}$$

Considering the asymptotic form of (3) as $r \to \infty$ gives

$$\lim_{|\underline{r}|\to\infty} \underline{\overline{E}}(\underline{r}) = \underline{J}_i \exp[i\underline{k}_i \underline{r}] + \frac{\exp(ik_0 r)}{r} \underline{\overline{J}}_s \left[\int_{V_1} \gamma \exp(-i\underline{k}_s \underline{r}_1) \underline{\overline{E}}(\underline{r}_1) d\underline{r}_1 + \int_{V_2} \gamma \exp(-i\underline{k}_s \underline{r}_2) \underline{\overline{E}}(\underline{r}_2) d\underline{r}_2 \right] \tag{14}$$

Therefore the scattering amplitude is expressed as

$$\underline{\overline{f}}(\underline{k}_s, \underline{k}_i) = \underline{\overline{J}}_s \left[\int_{V_1} \gamma \exp(-\underline{k}_s \underline{r}_1) \underline{\overline{E}}(\underline{r}_1) d\underline{r}_1 + \int_{V_2} \gamma \exp(-i\underline{k}_s \underline{r}_2) \underline{\overline{E}}(\underline{r}_2) d\underline{r}_2 \right] \tag{15}$$

3. METHOD OF SOLUTION

The field equation (3) is an integral equation with two singular kernels. In what follows, a method is applied which deals with the singularity analytically, leaving two integral equations with non singular kernels. Multiplying (3) by $\frac{\gamma}{k_0^2} \exp(-i\underline{k}_1 \underline{r})$ (where $\underline{k}_1$ is at present an arbitrary vector) and integrating throughout the volume of the scatterer V_1 gives

$$\begin{aligned} \frac{1}{k_0^2} \int_{V_1} \gamma \exp(-i\underline{k}_1 \underline{r}) \underline{\overline{E}}(\underline{r}) d\underline{r} = \underline{\overline{J}}_i \frac{1}{k_0^2} \int_{V_1} \gamma \exp[(\underline{k}_i - \underline{k}_1)\underline{r}] d\underline{r} + \frac{1}{k_0^2} \int_{V_1} d\underline{r} \gamma \exp(-i\underline{k}_1 \underline{r}) \int_{V_1} d\underline{r}_1 \gamma \underline{\overline{G}}(\underline{r}, \underline{r}_1) \underline{\overline{E}}(\underline{r}_1) + \\ + \frac{1}{k_0^2} \int_{V_1} d\underline{r} \gamma \exp(-i\underline{k}_1 \underline{r}) \int_{V_2} d\underline{r}_2 \gamma \underline{\overline{G}}(\underline{r}_1, \underline{r}_2) \underline{\overline{E}}(\underline{r}_2) \end{aligned} \tag{16}$$

The electric field inside the scatterer V_1 is expressed as the Fourier transform (xyz coordinate system)

$$\underline{\overline{E}}(\underline{r}_1) = \int d\underline{k}_2 \underline{\overline{C}}_1(\underline{k}_2) \exp(i\underline{k}_2 \underline{r}_1) \tag{17}$$

whereas inside the scatterer V2 is expressed as

$$\underline{\overline{E}}(\underline{r}_2) = \int d\underline{k}_2 \underline{\overline{C}}_2(\underline{k}_2) \exp(i\underline{k}_2 \underline{r}_2) \tag{18}$$

Substituting (17) and (18) into (16) gives

$$\int d\underline{k}_2 \begin{pmatrix} \underline{\overline{K}}^{aa}(\underline{k}_1, \underline{k}_2) & \underline{\overline{K}}^{ab}(\underline{k}_1, \underline{k}_2) \end{pmatrix} \begin{pmatrix} \underline{\overline{C}}_1(\underline{k}_2) \\ \underline{\overline{C}}_2(\underline{k}_2) \end{pmatrix} = \underline{\overline{J}}_i U_{V_1}(\underline{k}_1, \underline{k}_i) \tag{19}$$

where

$$U_{V_1}(\underline{k}_1,\underline{k}_2)=\frac{1}{k_0^2}\int_{V_1}\gamma\exp[-i(\underline{k}_1-\underline{k}_2)\underline{r}_1]\,d\underline{r}_1 \tag{20}$$

$$\overline{\underline{K}}^{aa}(\underline{k}_1,\underline{k}_2)=\overline{\underline{I}}U_{V_1}(\underline{k}_1,\underline{k}_2)-\frac{1}{k_0^2}\int_{V_1}d\underline{r}_1'\int_{V_1}d\underline{r}_1\,\gamma\exp(-i\underline{k}_1\underline{r}_1')\overline{\underline{G}}(\underline{r}_1',\underline{r}_1)\gamma\exp(i\underline{k}_2\underline{r}_1) \tag{21}$$

$$\overline{\underline{K}}^{ab}(\underline{k}_1,\underline{k}_2)=-\frac{1}{k_0^2}\int_{V_1}d\underline{r}_1\int_{V_2}d\underline{r}_2\,\gamma\exp(-i\underline{k}_1\underline{r}_1)\overline{\underline{G}}(\underline{r}_1,\underline{r}_2)\gamma\exp(i\underline{k}_2\underline{r}_2) \tag{22}$$

Now multiplying (3) by $\frac{\gamma}{k_0^2}\exp(-i\underline{k}_1\underline{r})$, integrating throughout the volume of the scatterer V_2 and substituting (17) and (18) into the resulting equation (which is similar to equation (16)) gives

$$\int d\underline{k}_2\begin{pmatrix}\overline{\underline{K}}^{ba}(\underline{k}_1,\underline{k}_2) & \overline{\underline{K}}^{bb}(\underline{k}_1,\underline{k}_2)\end{pmatrix}\begin{pmatrix}\overline{\underline{C}}_1(\underline{k}_2)\\ \overline{\underline{C}}_2(\underline{k}_2)\end{pmatrix}=\overline{\underline{J}}_i U_{V_2}(\underline{k}_1,\underline{k}_i) \tag{23}$$

where

$$U_{V_2}(\underline{k}_1,\underline{k}_2)=\frac{1}{k_0^2}\int_{V_2}d\underline{r}_2\,\gamma\,\exp[-i(\underline{k}_1-\underline{k}_2)\underline{r}_2] \tag{24}$$

$$\overline{\underline{K}}^{ba}(\underline{k}_1,\underline{k}_2)=-\frac{1}{k_0^2}\int_{V_2}d\underline{r}_2\int_{V_1}d\underline{r}_1\,\gamma\exp(-i\underline{k}_1\underline{r}_2)\overline{\underline{G}}(\underline{r}_2,\underline{r}_1)\gamma\exp(i\underline{k}_2\underline{r}_1) \tag{25}$$

$$\overline{\underline{K}}^{bb}(\underline{k}_1,\underline{k}_2)=\overline{\underline{I}}U_{V_2}(\underline{k}_1,\underline{k}_2)-\frac{1}{k_0^2}\int_{V_2}d\underline{r}_2'\int_{V_2}d\underline{r}_2\,\gamma\exp(-i\underline{k}_1\underline{r}_2')\overline{\underline{G}}(\underline{r}_2',\underline{r}_2)\gamma\exp(i\underline{k}_2\underline{r}_2) \tag{26}$$

Combining (19) and (23) gives

$$\int d\underline{k}_2\begin{pmatrix}\overline{\underline{K}}^{aa}(\underline{k}_1,\underline{k}_2) & \overline{\underline{K}}^{ab}(\underline{k}_1,\underline{k}_2)\\ \overline{\underline{K}}^{ba}(\underline{k}_1,\underline{k}_2) & \overline{\underline{K}}^{bb}(\underline{k}_1,\underline{k}_2)\end{pmatrix}\begin{pmatrix}\overline{\underline{C}}_1(\underline{k}_2)\\ \overline{\underline{C}}_2(\underline{k}_2)\end{pmatrix}=\begin{pmatrix}\overline{\underline{J}}_i U_{V_1}(\underline{k}_1,\underline{k}_i)\\ \overline{\underline{J}}_i U_{V_2}(\underline{k}_1,\underline{k}_i)\end{pmatrix} \tag{27}$$

Using the transform

$$\frac{\exp(ik_0|\underline{r}-\underline{r}'|)}{|\underline{r}-\underline{r}'|}=\frac{1}{2\pi^2}\lim_{\varepsilon'\to 0^+}\int\frac{d\underline{p}}{p^2-k_0^2-i\varepsilon'}\exp\left[i\underline{p}(\underline{r}-\underline{r}')\right] \tag{28}$$

and the dyadic relations

$$k_0^2\overline{\underline{G}}(\underline{r},\underline{r}')=-\overline{\underline{I}}\,4\pi\delta(\underline{r}-\underline{r}')+\nabla\times\{\nabla\times\overline{\underline{I}}\frac{\exp[ik_0|\underline{r}-\underline{r}'|]}{|\underline{r}-\underline{r}'|}\} \tag{29}$$

and

$$\nabla\times\nabla\times\overline{\underline{I}}G(\underline{r},\underline{r}')=\left(\nabla\nabla-\nabla^2\overline{\underline{I}}\right)G(\underline{r},\underline{r}') \tag{30}$$

equation (21) gives

$$\overline{\underline{K}}^{aa}(\underline{k}_1,\underline{k}_2)=\varepsilon\overline{\underline{I}}U_{V_1}(\underline{k}_1,\underline{k}_2)-\frac{1}{2\pi^2}\lim_{\varepsilon'\to 0^+}\int\frac{d\underline{p}}{p^2-k_0^2-i\varepsilon'}p^2(\overline{\underline{I}}-\hat{p}\hat{p})U_{V_1}(\underline{k}_1,\underline{p})U_{V_1}(\underline{p},\underline{k}_2) \tag{31}$$

If equations (8), (9), (10) and(11) are applied to (31) , the expression for $\overline{\underline{K}}^{aa}(\underline{k}_1,\underline{k}_2)$ is the following (see ref.[5,12] for relevant details)

$$\overline{\underline{K}}^{aa}(\underline{k}_1,\underline{k}_2)=\exp\left[-i\left(\underline{k}_1-\underline{k}_2\right)\underline{d}_1\right]\left[\overline{\underline{I}}\varepsilon(\varepsilon-1)a_1b_1c_1\frac{j_1\left(\left|\underline{\underline{K}}_{11}-\underline{\underline{K}}_{21}\right|_c\right)}{\left|\underline{\underline{K}}_{11}-\underline{\underline{K}}_{21}\right|_c}-\frac{(\varepsilon-1)^2}{2\pi^2}\left(a_1b_1c_1\right)^2\pi ik_0\int_0^1 dx_p\int_0^{2\pi}d\varphi_p\left[\overline{\underline{I}}-\hat{p}\hat{p}\right]\cdot\right.$$

$$\left.\sum_{\substack{n=0\\ n+m=\text{even}}}^{\infty}\sum_{m=0}^{\infty}(2n+3)(2m+3)\frac{j_{n+1}\left(\left|\underline{\underline{K}}_{11}\right|_c\right)}{\left|\underline{\underline{K}}_{11}\right|_c}\frac{j_{m+1}\left(\left|\underline{\underline{K}}_{21}\right|_c\right)}{\left|\underline{\underline{K}}_{21}\right|_c}\cdot\frac{j_{m_>+1}\left(k_0Y^{aa}\right)\cdot h_{m_<+1}\left(k_0Y^{aa}\right)}{(Y^{aa})^2}\cdot T_n^1\left(\hat{\underline{\underline{P}}}_1\cdot\hat{\underline{\underline{K}}}_{11}\right)T_m^1\left(\hat{\underline{\underline{P}}}_1\cdot\hat{\underline{\underline{K}}}_{21}\right)\right] \qquad (32)$$

where

$$m_>=\max\{m,n\},\qquad m_<=\min\{m,n\} \qquad (33)$$

$j_n(z)$ is the complex spherical Bessel function of order n, $T_n^1(x)$ is the Gegenbauer function,

$$h_n(x)=\left(\frac{\pi}{2x}\right)^{1/2}H^{(1)}_{n+\frac{1}{2}}(x) \qquad (34)$$

$$\underline{P}_1=p\left(\sqrt{1-x_p^2}\cos\varphi_p,\sqrt{1-x_p^2}\sin\varphi_p,x_p\right)\cdot\overline{\underline{A}}_1\equiv p\left(Z_x,Z_y,Z_z\right) \qquad (35)$$

$$x_a=\cos\left(\theta_a\right)\quad\text{for any a}$$

$$\underline{\underline{P}}_1=p\left(a_1Z_x,b_1Z_y,c_1Z_z\right) \qquad (36)$$

$$Y^{aa}=\sqrt{a_1^2Z_x^2+b_1^2Z_y^2+c_1^2Z_z^2} \qquad (37)$$

$$\hat{\underline{\underline{P}}}_1=\frac{1}{Y^{aa}}\left(a_1Z_x,b_1Z_y,c_1Z_z\right) \qquad (38)$$

the symbol $\left|\quad\right|_c$ is defined by the equation

$$\left|\underline{K}_1-\underline{K}_2\right|_c=\sqrt{K_1^2+K_2^2-2\underline{K}_1\cdot\underline{K}_2} \qquad (39)$$

where $\underline{K}_1=K_1\hat{K}_1$ (K_1 is complex) , $\underline{K}_2=K_2\hat{K}_2$ (K_2 is complex) and the symbol $\sqrt{z}=\sqrt{re^{i\varphi}}$ denotes the complex square root: $r^{\frac{1}{2}}e^{i\frac{\varphi}{2}}$.

$$\underline{K}_{11}=k_1\left(\sqrt{1-x_{k_1}^2}\cos\varphi_{k_1},\sqrt{1-x_{k_1}^2}\sin\varphi_{k_1},x_{k_1}\right)\cdot\overline{\underline{A}}_1\equiv k_1\left(H_x,H_y,H_z\right) \qquad (40)$$

(k_1 can take complex values)

$$\underline{\underline{K}}_{11}=k_1\left(a_1H_x,b_1H_y,c_1H_z\right) \qquad (41)$$

$$\left|\underline{\underline{K}}_{11}\right|_c = k_1\sqrt{a_1^2H_x^2 + b_1^2H_y^2 + c_1^2H_z^2} \tag{42}$$

$$\hat{\underline{\underline{K}}}_{11} = \frac{1}{\sqrt{a_1^2H_x^2 + b_1^2H_y^2 + c_1^2H_z^2}}\cdot\left(a_1H_x, b_1H_y, c_1H_z\right) \tag{43}$$

$$\underline{K}_{21} = k_2\left(\sqrt{1-x_{k_2}^2}\cos\varphi_{k_2}, \sqrt{1-x_{k_2}^2}\sin\varphi_{k_2}, x_{k_2}\right)\cdot\underline{\overline{A}}_1 \equiv k_2\left(\Lambda_x, \Lambda_y, \Lambda_z\right) \tag{44}$$

(k_2 can take complex values)

$$\underline{\underline{K}}_{21} = k_2\left(a_1\Lambda_x, b_1\Lambda_y, c_1\Lambda_z\right) \tag{45}$$

$$\left|\underline{\underline{K}}_{21}\right|_c = k_2\sqrt{a_1^2\Lambda_x^2 + b_1^2\Lambda_y^2 + c_1^2\Lambda_z^2} \tag{46}$$

$$\hat{\underline{\underline{K}}}_{21} = \frac{1}{\sqrt{a_1^2\Lambda_x^2 + b_1^2\Lambda_y^2 + c_1^2\Lambda_z^2}}\left(a_1\Lambda_x, b_1\Lambda_y, c_1\Lambda_z\right) \tag{47}$$

The dyadic $\underline{\overline{1}} - \hat{p}\hat{p}$ has the representation

$$\underline{\overline{1}} - \hat{p}\hat{p} = \begin{pmatrix} 1-y^2\cos^2\varphi_p & -y^2\cos\varphi_p\sin\varphi_p & -xy\cos\varphi_p \\ -y^2\cos\varphi_p\sin\varphi_p & 1-y^2\sin^2\varphi_p & -xy\sin\varphi_p \\ -xy\cos\varphi_p & -xy\sin\varphi_p & y^2 \end{pmatrix} \tag{48}$$

where $y^2 = 1-x^2$ and $x = \cos\theta_p$. The expression for the $\underline{\overline{K}}^{bb}$ element is similar to that of $\underline{\overline{K}}^{aa}$ (although not symmetric with respect to $\underline{\overline{K}}^{aa}$) and is given in Appendix A.

As far as the matrix elements $\underline{\overline{K}}^{ab}$ and $\underline{\overline{K}}^{ba}$ are considered, the only way to perform the p-integration is to apply the Cauchy's integral formula[11] and then to carry out numerical integration . The final form of the $\underline{\overline{K}}^{ab}$ matrix element is the following

$$\underline{\overline{K}}^{ab}\left(\underline{k}_1, \underline{k}_2\right) = -\frac{1}{2\pi^2}a_1b_1c_1a_2b_2c_2(\varepsilon-1)^2\exp\left[-i\left(\underline{k}_1\underline{d}_1 - \underline{k}_2\underline{d}_2\right)\right]\int_{-1}^{+1}dx_p\int_0^{2\pi}d\varphi_p\,\pi i\frac{k_0^2}{2}\left[\underline{\overline{1}} - \hat{p}\hat{p}\right]\exp\left[-i\left(\underline{d}_2 - \underline{d}_1\right)k_0\hat{p}\right]\cdot$$

$$\cdot\frac{j_1\left(\left|\underline{\underline{K}}_{11} - \underline{\underline{P}}_1\left(k_0\right)\right|_c\right)}{\left|\underline{\underline{K}}_{11} - \underline{\underline{P}}_1\left(k_0\right)\right|_c}\frac{j_1\left(\left|\underline{\underline{P}}_2\left(k_0\right) - \underline{\underline{K}}_{22}\right|_c\right)}{\left|\underline{\underline{P}}_2\left(k_0\right) - \underline{\underline{K}}_{22}\right|_c} - \frac{1}{2\pi^2}a_1b_1c_1a_2b_2c_2(\varepsilon-1)^2\exp\left[i\left(\underline{k}_1\underline{d}_1 - \underline{k}_2\underline{d}_2\right)\right]\cdot$$

$$\cdot\int_{-1}^{+1}dx_p\int_0^{2\pi}d\varphi_p\left[\int_0^{k_0-\delta}\frac{p^4dp}{p^2-k_0^2}\left[\underline{\overline{1}} - \hat{p}\hat{p}\right]\exp\left[-i\left(\underline{d}_2 - \underline{d}_1\right)p\hat{p}\right]\frac{j_1\left(\left|\underline{\underline{K}}_{11} - \underline{\underline{P}}_1\right|_c\right)}{\left|\underline{\underline{K}}_{11} - \underline{\underline{P}}_1\right|_c}\frac{j_1\left(\left|\underline{\underline{P}}_2 - \underline{\underline{K}}_{22}\right|_c\right)}{\left|\underline{\underline{P}}_2 - \underline{\underline{K}}_{22}\right|_c} + \right.$$

$$\left. + \int_{k_0+\delta}^{+\infty}\frac{p^4dp}{p^2-k_0^2}\left[\underline{\overline{1}} - \hat{p}\hat{p}\right]\exp\left[-i\left(\underline{d}_2 - \underline{d}_1\right)p\hat{p}\right]\frac{j_1\left(\left|\underline{\underline{K}}_{11} - \underline{\underline{P}}_1\right|_c\right)}{\left|\underline{\underline{K}}_{11} - \underline{\underline{P}}_1\right|_c}\frac{j_1\left(\left|\underline{\underline{P}}_2 - \underline{\underline{K}}_{22}\right|_c\right)}{\left|\underline{\underline{P}}_2 - \underline{\underline{K}}_{22}\right|_c}\right] \tag{49}$$

where the vectors $\underline{\underline{K}}_{11}$ $\underline{\underline{P}}_1$, $\underline{\underline{K}}_{22}$ and $\underline{\underline{P}}_2$ are given by (41), (36), (A.11) and (A.3) respectively. The real quantity δ used in the p-integration is positive and $\delta \to 0$.

Similarly the matrix element $\overline{\underline{K}}^{ba}$ is given by equation (A.14) (see Appendix A). The expressions for the matrix elements of the right part of (27) are to be found in Appendix B.

Substituting (17) and (18) into (15) gives

$$\overline{\underline{f}}(\underline{k}_s, \underline{k}_i) = k_0^2 \overline{\underline{J}}_s \int_{V_1} U_{V_1}(\underline{k}_s, \underline{k}_2) \overline{\underline{C}}_1(\underline{k}_2) d\underline{k}_2 + k_0^2 \overline{\underline{J}}_s \int_{V_2} U_{V_2}(\underline{k}_s, \underline{k}_2) \overline{\underline{C}}_2(\underline{k}_2) d\underline{k}_2 \qquad (50)$$

The integral equation (27) is reduced by quadrature into a set of linear equations (a matrix equation) which is solved by applying the Gauss-Jordan elimination algorithm. Thus the unknown dyadics $\overline{\underline{C}}_1(\underline{k}_2)$ and $\overline{\underline{C}}_2(\underline{k}_2)$ are calculated for a finite number of pivot vectors $\underline{k}_2$. Then the values of $\overline{\underline{C}}_1(\underline{k}_2)$ and $\overline{\underline{C}}_2(\underline{k}_2)$ are substituted into equation (50) and the scattering amplitude is obtained

4. CONVERGENCE ASPECTS

The case of scattering by one dielectric ellipsoid of complex index of refraction has already been treated in[5]. Therefore the matrix elements $\overline{\underline{K}}^{aa}(\underline{k}_1, \underline{k}_2)$ and $\overline{\underline{K}}^{bb}(\underline{k}_1, \underline{k}_2)$, which essentially describe the scattering of light by each one of the ellipsoids when considered separately, required relatively small programming modifications. On the contrary a substantial effort has been required for achievement of convergence in the calculation of the $\overline{\underline{K}}^{ab}(\underline{k}_1, \underline{k}_2)$ and $\overline{\underline{K}}^{ba}(\underline{k}_1, \underline{k}_2)$ matrix elements. Numerical integration in the complex plane was performed by using the Gauss integration formula of 16 points. Double precision complex arithmetic has been used in the numerical calculations..

A list of typical model input parameters which have been used as an example for the calculation of $\overline{\underline{K}}^{ab}(\underline{k}_1, \underline{k}_2)$ and $\overline{\underline{K}}^{ba}(\underline{k}_1, \underline{k}_2)$ are given together with typical values of the number of integration subintervals necessary for convergence.

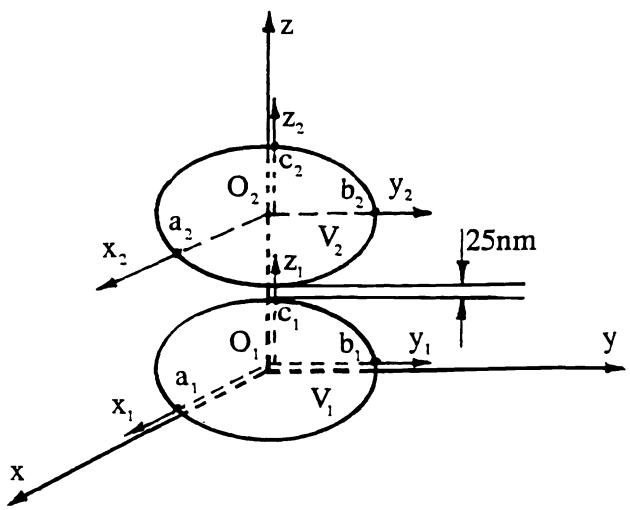

Fig2. Relative position of the RBCs considered in the $\overline{\underline{K}}^{ab}(\underline{k}_1, \underline{k}_2)$ and $\overline{\underline{K}}^{ba}(\underline{k}_1, \underline{k}_2)$ convergence example

Incident wavelength (in vacuum): $\lambda = 0.633 \cdot 10^{-6}$ m (HeNe laser beam)

Refractive index of plasma: $n_p = 1.33$

Incident wave number (in plasma): $k_0 = 13.20 \cdot 10^6$ m^{-1}

Relative (to plasma) complex refractive index of a RBC [4,14,15]: $n_0 = \sqrt{\varepsilon} = 1.04 + i\,10^{-4}$

Complex wave number inside each RBC: $k_1 = k_2 = (13.729 \cdot 10^6 + i\,1320.1)$ m^{-1}

Semiaxes of the elipsoids (spheroidal case)

$a_1 = a_2 = 4 \cdot 10^{-6}$ m (4μm)

$b_1 = b_2 = 4 \cdot 10^{-6}$ m (4μm)

$c_1 = c_2 = 1.3 \cdot 10^{-6}$ m (1.3μm)

The values of c_1, c_2 were chosen so that the volume of each spheroid corresponds to the volume of a typical RBC (around 87 μm^3).

Position vector $\underline{d}_1$: $\left\{ \begin{array}{c} d_1 = 0 \text{ m} \quad (0\mu m) \\ x_{d_1} = \cos(\theta_{d_1}) = 1 \\ \varphi_{d_1} = 0 \text{ rad} \end{array} \right\}$

Position vector $\underline{d}_2$: $\left\{ \begin{array}{c} d_2 = 2.625 \cdot 10^{-6} \text{ m} \quad (2.625\,\mu m) \\ x_{d_2} = \cos(\theta_{d_2}) = 1 \\ \varphi_{d2} = 0 \text{ rad} \end{array} \right\}$

The relative position of the RBCs is shown in Fig.2.

In order to achieve convergence in the calculation of the $\overline{\underline{K}}^{ab}(\underline{k}_1, \underline{k}_2)$ and $\overline{\underline{K}}^{ba}(\underline{k}_1, \underline{k}_2)$ elements (given by equations (49) and (A.14)) each interval on the p-axis of length k_0 should be divided into at least three subintervals and p-integration should be performed at each one of them. The upper bound of the integral, which is theoretically infinity, can be truncated to the value of 30 k_0. The x_p integration interval should be divided into at least 10 subintervals whereas the φ_p integration interval should be divided into at least 20 subintervals. A good value for δ has been shown to be $\delta = 10^{-6}$. Routines for the solution of the complete matrix system are being currently developed.

From what has been mentioned so far it is clear that the computing cost of the model is substantially high. In order to make execution faster, parallel processing is being undergone on various Silicon Graphics power systems. It is expected that the use of parallel processing systems would decrease the execution time drastically.

5. CONCLUSIONS

A new mathematical model simulating light scattering by a system of two adjacent red blood cells has been presented. The model is based on the general Fredholm integral theory and overcomes many limitations of relevant approximate theories. The main drawback of the model, which is the relatively high computing time required, is being lessend by the use of parallel processing systems. Prospective areas of application include laser angioplasty, laser RBC aggregometry, flow cytometry as well as non biomedical areas such as astrophysics and atmospherical physics.

APPENDIX A

Expressions for the $\overline{\underline{K}}^{bb}(\underline{k}_1,\underline{k}_2)$ and $\overline{\underline{K}}^{ba}(\underline{k}_1,\underline{k}_2)$ matrix elements

The matrix element $\overline{\underline{K}}^{bb}(\underline{k}_1,\underline{k}_2)$ has the following expression

$$\overline{\underline{K}}^{bb}(\underline{k}_1,\underline{k}_2)=\exp\left[-i(\underline{k}_1-\underline{k}_2)\underline{d}_2\right]\left[\bar{\underline{1}}\varepsilon(\varepsilon-1)a_2b_2c_2\frac{j_1\left(\left|\underline{\underline{K}}_{12}-\underline{\underline{K}}_{22}\right|_c\right)}{\left|\underline{\underline{K}}_{12}-\underline{\underline{K}}_{22}\right|_c}-\right.$$

$$-\frac{(\varepsilon-1)^2}{2\pi^2}\left(a_2b_2c_2\right)^2\pi ik_0\int_0^1 dx_p\int_0^{2\pi}d\varphi_p\left[\bar{\underline{1}}-\hat{p}\hat{p}\right]\sum_{\substack{n=0\\ n+m=\\ even}}^{\infty}\sum_{m=0}^{\infty}(2n+3)(2m+3)\frac{j_{n+1}\left(\left|\underline{\underline{K}}_{12}\right|_c\right)}{\left|\underline{\underline{K}}_{12}\right|_c}\frac{j_{m+1}\left(\left|\underline{\underline{K}}_{22}\right|_c\right)}{\left|\underline{\underline{K}}_{22}\right|_c}\frac{j_{m_>+1}\left(k_0Y^{bb}\right)h_{m_<+1}\left(k_0Y^{bb}\right)}{\left(Y^{bb}\right)^2}\cdot$$

$$\left.\cdot T_n^1\left(\hat{\underline{\underline{P}}}_2\cdot\hat{\underline{\underline{K}}}_{12}\right)T_m^1\left(\hat{\underline{\underline{P}}}_2\cdot\hat{\underline{\underline{K}}}_{22}\right)\right] \tag{A.1}$$

where

$$m_>=\max\{m,n\},\quad m_<=\min\{m,n\}$$

$$\underline{P}_2=p\left(\sqrt{1-x_p^2}\cos\varphi_p,\sqrt{1-x_p^2}\sin\varphi_p,x_p\right)\cdot\underline{\overline{A}}_2\equiv p\left(\Xi_x,\Xi_y,\Xi_z\right) \tag{A.2}$$

$$\underline{\underline{P}}_2=p\left(a_2\Xi_x,b_2\Xi_y,c_2\Xi_z\right) \tag{A.3}$$

$$Y^{bb}=\sqrt{a_2^2\Xi_x^2+b_2^2\Xi_y^2+c_2^2\Xi_z^2} \tag{A.4}$$

$$\hat{\underline{\underline{P}}}_2=\frac{1}{Y^{bb}}\left(a_2\Xi_x,b_2\Xi_y,c_2\Xi_z\right) \tag{A.5}$$

$$\underline{K}_{12}=k_1\left(\sqrt{1-x_{k_1}^2}\cos\varphi_{k_1},\sqrt{1-x_{k_1}^2}\sin\varphi_{k_1},x_{k_1}\right)\cdot\underline{\overline{A}}_2\equiv k_1\left(M_x,M_y,M_z\right) \tag{A.6}$$

$$\underline{\underline{K}}_{12}=k_1\left(a_2M_x,b_2M_y,c_2M_z\right) \tag{A.7}$$

$$\left|\underline{\underline{K}}_{12}\right|_c=k_1\sqrt{a_2^2M_x^2+b_2^2M_y^2+c_2^2M_z^2} \tag{A.8}$$

$$\hat{\underline{\underline{K}}}_{12}=\frac{1}{\sqrt{a_2^2M_x^2+b_2^2M_y^2+c_2^2M_z^2}}\left(a_2M_x,b_2M_y,c_2M_z\right) \tag{A.9}$$

$$\underline{K}_{22}=k_2\left(\sqrt{1-x_{k_1}^2}\cos\varphi_{k_2},\sqrt{1-x_{k_2}^2}\sin\varphi_{k_2},x_{k_2}\right)\cdot\underline{\overline{A}}_2\equiv k_2\left(T_x,T_y,T_z\right) \tag{A.10}$$

$$\underline{\underline{K}}_{22}=k_2\left(a_2T_x,b_2T_y,c_zT_z\right) \tag{A.11}$$

$$\left|\underline{\underline{K}}_{22}\right|_c=k_2\sqrt{a_2^2T_x^2+b_2^2T_y^2+c_2^2T_z^2} \tag{A.12}$$

$$\underline{\underline{\hat{K}}}_{22} = \frac{1}{\sqrt{a_2^2T_x^2 + b_2^2T_y^2 + c_2^2T_z^2}}\left(a_2T_x, b_2T_y, c_2T_z\right) \tag{A.13}$$

The expression for the matrix element $\underline{\underline{\overline{K}}}^{ba}\left(\underline{k}_1, \underline{k}_2\right)$ is the following

$$\underline{\underline{\overline{K}}}^{ba}\left(\underline{k}_1, \underline{k}_2\right) = -\frac{1}{2\pi^2}a_1b_1c_1a_2b_2c_2(\varepsilon-1)^2\exp\left[-i\left(\underline{k}_1\underline{d}_2 - \underline{k}_2\underline{d}_1\right)\right]\int_{-1}^{+1}dx_p\int_0^{2\pi}d\varphi_p\,\pi i\frac{k_0^3}{2}\left[\underline{\underline{I}} - \hat{p}\hat{p}\right]\exp\left[i\left(\underline{d}_2 - \underline{d}_1\right)\underline{p}\right]\cdot$$

$$\cdot\frac{j_1\left(\left|\underline{\underline{K}}_{12} - \underline{\underline{P}}_2\left(k_0\right)\right|_c\right)}{\left|\underline{\underline{K}}_{12} - \underline{\underline{P}}_2\left(k_0\right)\right|_c}\frac{j_1\left(\left|\underline{\underline{P}}_1\left(k_0\right) - \underline{\underline{K}}_{21}\right|_c\right)}{\left|\underline{\underline{P}}_1\left(k_0\right) - \underline{\underline{K}}_{21}\right|_c} - \frac{1}{2\pi^2}a_1b_1c_1a_2b_2c_2(\varepsilon-1)^2\exp\left[-i\left(\underline{k}_1\underline{d}_2 - \underline{k}_2\underline{d}_1\right)\right]\int_{-1}^{+1}dx_p\int_0^{2\pi}d\varphi_p\left[\int_0^{k_0-\delta}\frac{p^4dp}{p^2-k_0^2}\cdot\right.$$

$$\left.\cdot\left[\underline{\underline{I}} - \hat{p}\hat{p}\right]\exp\left[i\left(\underline{d}_2 - \underline{d}_1\right)\underline{p}\right]\frac{j_1\left(\left|\underline{\underline{K}}_{12} - \underline{\underline{P}}_2\right|_c\right)}{\left|\underline{\underline{K}}_{12} - \underline{\underline{P}}_2\right|_c}\frac{j_1\left(\left|\underline{\underline{P}}_1 - \underline{\underline{K}}_{21}\right|_c\right)}{\left|\underline{\underline{P}}_1 - \underline{\underline{K}}_{21}\right|_c} + \int_{k_0+\delta}^{\infty}\frac{p^4dp}{p^2-k_0^2}\left[\underline{\underline{I}} - \hat{p}\hat{p}\right]\exp\left[i\left(\underline{d}_2 - \underline{d}_1\right)\underline{p}\right]\frac{j_1\left(\left|\underline{\underline{K}}_{12} - \underline{\underline{P}}_2\right|_c\right)}{\left|\underline{\underline{K}}_{12} - \underline{\underline{P}}_2\right|_c}\cdot\right.$$

$$\left.\cdot\frac{j_1\left(\left|\underline{\underline{P}}_1 - \underline{\underline{K}}_{21}\right|_c\right)}{\left|\underline{\underline{P}}_1 - \underline{\underline{K}}_{21}\right|_c}\right] \tag{A.14}$$

where the vectors $\underline{\underline{K}}_{12}$, $\underline{\underline{P}}_1$, $\underline{\underline{K}}_{21}$ and $\underline{\underline{P}}_2$ are given by equations (A.7), (36), (45) and (A.3) respectively.

APPENDIX B

Expressions for the matrix elements of the right part of equation (27)

The elements of the right part of the matrix equation (27) take the following form

$$\underline{\underline{J}}_i U_{V_1}\left(\underline{k}_1, \underline{k}_i\right) = \left(\underline{\underline{I}} - \hat{k}_i\hat{k}_i\right)\exp\left[-i\left(\underline{k}_1 - \underline{k}_i\right)\underline{d}_1\right]a_1b_1c_1(\varepsilon-1)\frac{j_1\left(\left|\underline{\underline{K}}_{11} - \underline{\underline{K}}_{i1}\right|_c\right)}{\left|\underline{\underline{K}}_{11} - \underline{\underline{K}}_{i1}\right|_c} \tag{B.1}$$

where $\underline{\underline{K}}_{11}$ is given by (41) and

$$\underline{K}_{i1} = k_0\left(\sqrt{1-x_{k_i}^2}\cos\varphi_{k_i}, \sqrt{1-x_{k_i}^2}\sin\varphi_{k_i}, x_{k_i}\right)\cdot\underline{\underline{A}}_1 \equiv \left(\Sigma_x, \Sigma_y, \Sigma_z\right) \tag{B.2}$$

$$\underline{\underline{K}}_{i1} = \left(a_1\Sigma_x, b_1\Sigma_y, c_1\Sigma_z\right) \tag{B.3}$$

$$\underline{\underline{J}}_i U_{V_2}\left(\underline{k}_1, \underline{k}_i\right) = \left(\underline{\underline{I}} - \hat{k}_i\hat{k}_i\right)\exp\left[-i\left(\underline{k}_1 - \underline{k}_i\right)\underline{d}_2\right]a_2b_2c_2(\varepsilon-1)\frac{j_1\left(\left|\underline{\underline{K}}_{12} - \underline{\underline{K}}_{i2}\right|_c\right)}{\left|\underline{\underline{K}}_{12} - \underline{\underline{K}}_{i2}\right|_c} \tag{B.4}$$

where $\underline{\underline{K}}_{12}$ is given by (A.7) and

$$\underline{K}_{i2} = k_0\left(\sqrt{1-x_{k_i}^2}\cos\varphi_{k_i}, \sqrt{1-x_{k_i}^2}\sin\varphi_{k_i}, x_{k_i}\right)\cdot\underline{\bar{A}}_2 \equiv \left(\Omega_x, \Omega_y, \Omega_z\right) \tag{B.5}$$

$$\underline{\underline{K}}_{i2} = \left(a_2\Omega_x, b_2\Omega_y, c_2\Omega_z\right) \tag{B.6}$$

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