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Compendium of Formulas for the Structural Vibration Frequency Analysis of Beams

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(Professional development hours = 3)

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1. INTRODUCTION

Fundamental natural frequency is an important parameter in structural dynamics analysis and applications. A number of simple formulas for the computation of the fundamental natural frequency of beam vibration under a variety of boundary conditions and loading are presented. Some are approximate formulas and some are exact. An estimate of the error is given for the approximate formulas whenever possible. The formulas presented here are simple enough that the fundamental frequency could be obtained by hand calculation without recourse to computers or finite element analysis. The formulas are demonstrated with nine worked out numerical examples. [Fundamental natural frequency is also referred as fundamental frequency. If a structure has more than one natural frequency the lowest frequency is the fundamental frequency.]

2. SINGLE-DEGREE-OF-FREEDOM SYSTEM

Beams have infinite degrees-of-freedom. However beams may be modeled as single-degree-freedom (SDOF) systems under certain conditions. This forms the basis for some of the formulas presented here. So we present in this section natural frequency computation of SDOF systems.

2.1 Frequency Analysis of Single-Degree-of-Freedom Systems

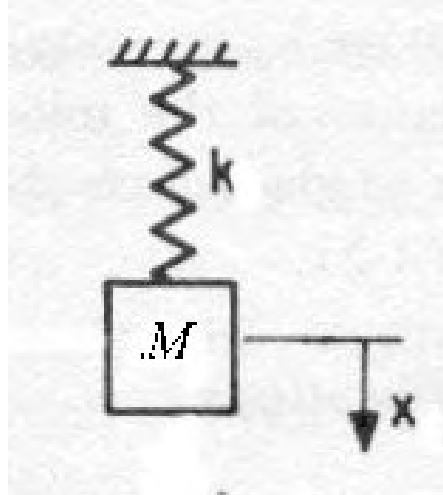


Figure 1: Single-Degree-of-Freedom System Model for Frequency Analysis

k = stiffness,
 M = mass, and
 x = displacement of the mass

Natural frequency of the SDOF system in radians per second is given by

$$\omega = \sqrt{\frac{k}{M}} \quad (2.1)$$

Natural frequency in cycles per second is given by

$$f = \omega/2\pi \quad (2.2)$$

Period of the SDOF system in seconds is given by

$$T = \frac{1}{f} \quad (2.3)$$

2.2 Static Deflection Approach

Equation (2.1) may be rewritten as

$$\omega = \sqrt{\frac{k}{M}} = \sqrt{\frac{k}{(W/g)}} = \sqrt{\frac{kg}{W}} = \sqrt{\frac{g}{\Delta}} \quad (2.4)$$

In the above equation we have substituted

$M = W/g$ (mass = weight/acceleration due to gravity)

and

$\Delta = W/k$ (Static deflection due to weight = weight/stiffness)

Thus the formula for natural frequency in radians per seconds is,

$$\omega = \sqrt{\frac{g}{\Delta}} \quad (2.5)$$

3. BEAM WITH A CONCENTRATED MASS

3.1 Simple Supported Beam

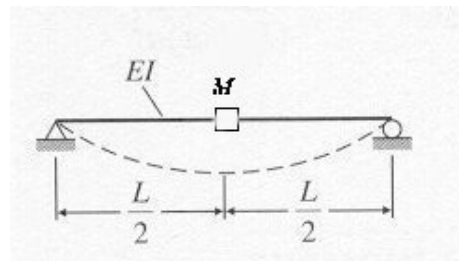


Figure 2: Simple Supported Beam with Concentrated Mass at Mid-span

Consider a uniform beam with both ends simple supported carrying a concentrated mass at mid-span (Figure 2). Distributed mass along the span (for example mass of the beam) is much smaller than the concentrated mass that we ignore the distributed mass in the frequency analysis. [What we mean by "much smaller" is discussed later in Section 3.]

Example of a concentrated mass is the mass of a machine placed at mid-span of the beam.

So, with only the single concentrated mass this beam can be modeled as an equivalent single-degree-of-freedom (SDOF) system as shown in Figure 1. Equivalent stiffness for this case is,

$$k = \frac{48EI}{L^3}$$

Mass of the SDOF is equal to the concentrated mass on the beam. How did we get the stiffness? We use the definition of stiffness. Stiffness is the force required to produce unit deflection. Alternately, stiffness is equal to force divided by deflection.

In order to compute the equivalent stiffness of the simple supported beam, apply a concentrated force (F) at mid-span and determine the deflection (Δ) at the mid-span.

$$\Delta = \frac{FL^3}{48EI}$$

Where E = Young's modulus of elasticity,
I = Moment of inertia of beam cross section, and
L = Span.

You may find this formula for static beam deflection in many structural engineering and engineering mechanics textbooks.

By definition, stiffness $k = \frac{\text{Force}}{\text{Deflection}} = \frac{F}{\Delta} = \frac{48EI}{L^3}$

Using eqn. (2.1), natural frequency ω of a uniform simple supported beam with a heavy concentrated mass at mid-span is given by

$$\omega^2 = \frac{k}{M} = \frac{48EI}{ML^3} \quad (3.1)$$

3.2 Beams with Other Boundary Conditions

The frequency calculation approach used in Section 3.1 is applicable to beams with other boundary conditions also. As long as the beam is uniform and the distributed mass along the span is much smaller than the concentrated mass at mid-span, eqn. (2.1) is applicable. Equivalent stiffness (k) for various boundary conditions are given in Table 1.

	Boundary Conditions	Equivalent Stiffness (k)
1.	Both ends simply supported	$\frac{48EI}{L^3}$
2.	Both ends clamped	$\frac{192EI}{L^3}$
3.	One end clamped and the other end free (cantilever)	$\frac{3EI}{L^3}$
4.	One end clamped and the other end simply supported (propped cantilever)	$\frac{768EI}{7L^3}$

Equivalent Stiffness
Table 1

3.3 Static Deflection Approach

The static deflection approach discussed for SDOF in Section 2.2 is applicable for beams with concentrated mass also.

Using eqn. (2.5)

$$\omega^2 = \frac{g}{\Delta}$$

Static deflection due to concentrated weight, $\Delta = W/k$

where, k is the beam stiffness provided in Section 3.2. This will provide same natural frequency as Section 3.2. For example, if we use this formula for beams with both ends simple supported,

$$\Delta = \frac{WL^3}{48EI}$$

So,

$$\omega^2 = \frac{g}{\Delta} = \frac{48EI}{ML^3} \quad \text{because } m = W/g$$

As we can see this equation is identical to eqn. (3.1). If the approaches presented in Sections 3.2 and 3.3 give the same results, what is the purpose of Section 3.3? Some engineers prefer to deal in terms of deflections than stiffness.

3.4 Non Central Mass

The concentrated mass is at a distance "a" from the left and a distance "b" from the right end. A simple supported beam is shown in Figure 3. Equivalent stiffness values (k) for various boundary conditions are given below.

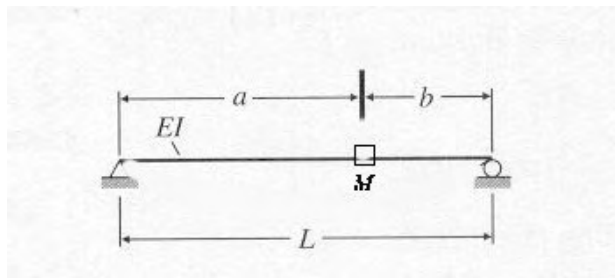


Figure 3: Simple Supported Beam with a Non-central Concentrated Mass

Both ends simple supported

$$k = \frac{3EIL}{a^2b^2}$$

Both ends clamped

$$k = \frac{3EI(a+b)^3}{a^3b^3}$$

3.5 Approximations and Accuracy

The primary approximation of the method presented in Section 3 is that the distributed mass of the beam (for example, self mass) is ignored. Because some mass is ignored, this approximation results in a higher value for the fundamental natural frequency. Error

introduced by this approximation reduces as the ratio of concentrated mass to distributed mass increases. Error is about 25% when the ratio is 1. This 25% is an approximate estimate and the actual error not only depends on the mass ratio but also on boundary conditions. If the mass ratio is 2, error is approximately 15%.

Numerical Example-1

Problem:

Compute the fundamental natural frequency of a 200" span beam carrying a central concentrated weight of 7000 pounds. Distributed weight along the span is very small compared to the concentrated weight that it may be ignored in frequency calculations. Both ends of the beam are simple supported. Young's modulus of elasticity (E) = 30×10^6 psi, Moment of inertia (I) = 400 inch^4 .

Solution:

Span = 200"

Concentrated weight at mid-span (W) = 7000 lb

So,

Concentrated mass at mid-span (M) = $W/g = (7000/386.4) = 18.12 \text{ lb-sec}^2/\text{inch}$
where $g = 386.4 \text{ inch/sec}^2$

From Table 1,

$$k = \frac{48EI}{L^3} = [48 \times (30 \times 10^6) \times 400] / 200^3 \\ = (72 \times 10^3) \text{ lb/inch}$$

From eqns. (2.1), (2.2) and (2.3),

Fundamental natural frequency in radians per second,

$$\omega = \sqrt{\frac{k}{M}} = \sqrt{\frac{(72 \times 10^3)}{18.12}} = 63 \text{ rad/sec}$$

Fundamental natural frequency in cycles per second,

$$f = \frac{\omega}{2\pi} = \frac{63}{2\pi} = 10 \text{ cps}$$

$$\text{Period } T = \frac{1}{f} = (1/10) = 0.1 \text{ secs}$$

Numerical Example-2

Problem:

Same beam as Numerical Example-1 but both ends of the beam are clamped (fixed).
Compute the fundamental natural frequency.

Solution:

Span = 200"

Concentrated weight at mid-span (W) = 7000 lb

So,

Concentrated mass at mid-span (M) = $W/g = (7000/386.4) = 18.12 \text{ lb-sec}^2/\text{inch}$
where $g = 386.4 \text{ inch/sec}^2$

From Table 1,

$$k = \frac{192EI}{L^3} = [192 \times (30 \times 10^6) \times 400] / 200^3$$

$$= (288 \times 10^3) \text{ lb/inch}$$

From eqns. (2.1), (2.2) and (2.3),

Fundamental natural frequency in radians per second,

$$\omega = \sqrt{\frac{k}{M}} = \sqrt{\frac{(288 \times 10^3)}{18.12}} = 126 \text{ rad/sec}$$

Fundamental natural frequency in cycles per second,

$$f = \frac{\omega}{2\pi} = \frac{126}{2\pi} = 20 \text{ cps}$$

$$\text{Period } T = \frac{1}{f} = (1/20) = 0.05 \text{ secs}$$

Numerical Example-3

Problem:

Compute the fundamental natural frequency of the beam described in Numerical Example-1, using the static deflection approach described in Section 3.3.

Solution:

Static deflection due to the concentrated weight = $\Delta = W/k$

For a beam simply supported at both ends,

$$k = \frac{48EI}{L^3} \quad (\text{see Section 3.2.1})$$

So,

$$\Delta = \frac{WL^3}{48EI} = \frac{7000 \times (200)^3}{48 \times (30 \times 10^6) \times 400} = 0.0972 \text{ inch}$$

From eqn. (2.5),

$$\omega = \sqrt{\frac{g}{\Delta}} = \sqrt{\frac{386.4}{0.0972}} = 63 \text{ rad/sec}$$

4. BEAM WITH UNIFORMLY DISTRIBUTED MASS

A beam with uniformly distributed mass has infinite natural frequencies. Natural frequencies are arranged in ascending order. The lowest natural frequency is called the fundamental frequency or the fundamental natural frequency. The formula presented in Section 4 is "exact"; no approximation is made.

The n-th natural frequency ω_n is given by

$$\omega_n^2 = \beta_n^4 \frac{EI}{(m/L)} \quad (4.1)$$

where

E = Young's modulus of elasticity,
I = moment of inertia of cross section,
L = length of beam (span), and
m = total distributed mass.

β_1, β_2 and β_3 for various boundary conditions are listed in Table 2. Refer to Structural Dynamics textbooks on to get higher values of β by solving transcendental equations. That is beyond the scope of the course.

	Boundary Conditions	β values
1.	Both ends simply supported	$\beta_1 L = \pi$ $\beta_2 L = 2\pi$ $\beta_3 L = 3\pi$
2.	Both ends clamped	$\beta_1 L = 4.7300$ $\beta_2 L = 7.8532$ $\beta_3 L = 10.9956$
3.	One end clamped and the other end free (cantilever)	$\beta_1 L = 1.8751$ $\beta_2 L = 4.6941$ $\beta_3 L = 7.8548$
4.	One end clamped and the other end simply supported (propped cantilever)	$\beta_1 L = 3.9266$ $\beta_2 L = 7.0686$ $\beta_3 L = 10.2102$

Beam Natural Frequency Computation
Table 2

Numerical Example-4

Problem:

Compute the fundamental frequency of a 200" span beam carrying a uniformly distributed weight of 2000 pounds (total weight w). There are no concentrated weights. Both ends of the beam are simply supported. Young's modulus of elasticity (E) = 30×10^6 psi, Moment of inertia (I) = 400 inch^4 .

Solution:

Using eqn. (4.1), the fundamental natural frequency ω_1 is given by

$$\omega_1^2 = \beta_1^4 \frac{EI}{(m/L)}$$

In our problem,

$$\beta_1 L = \pi \text{ (from Table 2)}$$

So,

$$\beta_1 = \pi/L = (3.14/200) = 0.0157$$

$$E = 30 \times 10^6 \text{ psi}$$

$$I = 400 \text{ inch}^4$$

$$m = w/g = 2000/386.4 = 5.176 \text{ lb-sec}^2/\text{inch}$$

$$L = 200 \text{ inch}$$

Substitution of these values into eqn. (4.1) yields,

$$\omega_1^2 = 28190$$

$$\omega_1 = 168 \text{ rad/sec}$$

Using eqns. (2.2) and (2.3),

Fundamental natural frequency in cycles per second,

$$f = \frac{\omega}{2\pi} = \frac{168}{2\pi} = 26.8 \text{ cps}$$

$$\text{Period } T = \frac{1}{f} = (1/26.8) = 0.037 \text{ secs}$$

Numerical Example-5

Problem:

Same beam as Numerical Example-4 but both ends of the beam are clamped (fixed).
Compute the fundamental natural frequency.

Solution:

Using eqn. (4.1), the fundamental natural frequency ω_1 is given by

$$\omega_1^2 = \beta_1^4 \frac{EI}{(m/L)}$$

In our problem,

$$\beta_1 L = 4.73 \text{ (from Table 1)}$$

So,

$$\beta_1 = 4.73/L = (4.73/200) = 0.02365$$

$$E = 30 \times 10^6 \text{ psi}$$

$$I = 400 \text{ inch}^4$$

$$m = w/g = 2000/386.4 = 5.176 \text{ lb-sec}^2/\text{inch}$$

$$L = 200 \text{ inch}$$

Substitution of these values into eqn. (4.1) yields,

$$\omega_1^2 = 145009$$

$$\omega_1 = 381 \text{ rad/sec}$$

Using eqns. (2.2) and (2.3),

Fundamental natural frequency in cycles per second,

$$f = \frac{\omega}{2\pi} = \frac{381}{2\pi} = 60.7 \text{ cps}$$

$$\text{Period } T = \frac{1}{f} = (1/60.7) = 0.0165 \text{ secs}$$

5. BEAM WITH UNIFORMLY DISTRIBUTED MASS AND A CONCENTRATED MASS

Wachel and Bates [1] provide an approximate formula for the fundamental frequency of beams with uniformly distributed mass and a concentrated mass.

$$\omega^2 = \omega_u^2 / (1 + cR) \quad (5.1)$$

where

ω_u = fundamental natural frequency of the beam with uniformly distributed mass only (no concentrated mass),

R = ratio of concentrated mass to total uniformly distributed mass (= M/m)

c = correction factor (see Table 2)

M = concentrated mass, and

m = total uniformly distributed mass.

Boundary Conditions	Location of Concentrated Mass	Correction Factor (c)
Both ends simply supported	Mid-span	2.0
Both ends clamped	Mid-span	2.7
One end clamped, other end free (cantilever)	Free end	3.9
One end clamped, other end simply supported (propped cantilever)	Mid-span	2.3

Wachel and Bates Correction Factor
Table 3

Wachel and Bates [2] do not provide an error estimate for the formula. Error in the formula increases with the ratio (M/m). The formula may give good approximate results for (M/m) = 1.

Numerical Example-6

Problem:

Compute the fundamental natural frequency of a 200" span beam carrying a uniformly distributed weight of 2000 pounds (total weight over the span) and a concentrated weight of 1000 pounds at mid-span. Both ends of the beam are simply supported. Young's modulus of elasticity (E) = 30×10^6 psi, Moment of inertia (I) = 400 inch⁴.

Solution:

We will use eqn. (5.1) to compute the fundamental natural frequency.

$$\omega^2 = \omega_u^2 / (1 + cR)$$

Here ω_u is the fundamental natural frequency of the beam with uniformly distributed mass only (no concentrated mass). We have already computed this value in Numerical Example-4.

$$\omega_u^2 = 28190$$

$$\begin{aligned} R = (M/m) &= (\text{concentrated mass} / \text{total uniformly distributed mass}) \\ &= (\text{concentrated weight} / \text{total uniformly distributed weight}) \\ &= (1000 / 2000) = 0.5 \end{aligned}$$

c = correction factor = 2.0 (from Table 2)

Substituting these values in eqn. (5.1), we get

$$\omega^2 = 14095$$

$$\omega = 118.7 \text{ rad/sec}$$

Using eqns. (2.2) and (2.3),

Fundamental natural frequency in cycles per second,

$$f = \frac{\omega}{2\pi} = \frac{118.7}{2\pi} = 18.9 \text{ cps}$$

$$\text{Period } T = \frac{1}{f} = (1/18.9) = 0.053 \text{ secs}$$

Numerical Example-7

Problem:

Same beam as Numerical Example-6 but both ends of the beam are clamped (fixed). Compute the fundamental natural frequency.

Solution:

We will use eqn. (5.1) to compute the fundamental natural frequency.

$$\omega^2 = \omega_u^2 / (1 + cR)$$

Here ω_u is the fundamental natural frequency of the beam with uniformly distributed mass only (no concentrated mass). We have already computed this value in Numerical Example-5.

$$\omega_u^2 = 145009$$

$$R = (M/m) = (1000 / 2000) = 0.5$$

c = correction factor = 2.7 (from Table 2)

Substituting these values in eqn. (5.1), we get

$$\omega^2 = 61706$$

$$\omega = 248.4 \text{ rad/sec}$$

Using eqns. (2.2) and (2.3),

Fundamental natural frequency in cycles per second,

$$f = \frac{\omega}{2\pi} = \frac{248.4}{2\pi} = 39.5 \text{ cps}$$

$$\text{Period } T = \frac{1}{f} = (1/39.5) = 0.025 \text{ secs}$$

6. BEAMS WITH MULTIPLE CONCENTRATED MASSES

Dunkerley's method may be used to compute the fundamental frequency of beams with multiple concentrated masses (and no distributed mass).

Consider a beam with n concentrated masses $M_1, M_2, M_3, \dots M_n$. (Figure 4). This is an n -degrees-of-freedom system and it has n natural frequencies and the lowest frequency is the fundamental natural frequency. The following procedure is used to compute the fundamental natural frequency.

Remove all masses except the first mass M_1 . Compute the natural frequency using the method described in Section 3. Let this frequency be denoted by Ω_1 . Next remove all masses except the second mass M_2 and compute the natural frequency Ω_2 . Repeat it for all n masses.

Fundamental natural frequency of the beam (ω) is given by Dunkerley's equation

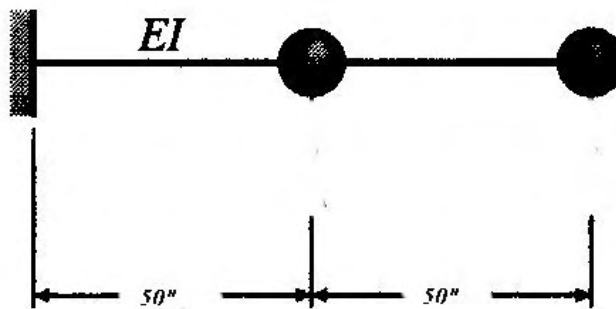
$$\frac{1}{\omega^2} = \frac{1}{\Omega_1^2} + \frac{1}{\Omega_2^2} + \dots + \frac{1}{\Omega_n^2} \quad (6.1)$$

Dunkerley's equation is a lower bound to the exact solution. Accuracy of results depends on beam boundary conditions, number of masses and relative values of the masses. In a couple of problems this writer has solved and compared with other methods, error was less than 5%. This does not in anyway mean that we will get such accuracy for all problems.

Numerical Example-8

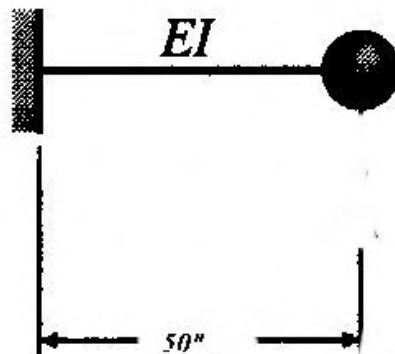
Problem:

A 100" span cantilever beam carries two concentrated weights, 2000 pounds each, at 50" and 100" from the fixed end. Distributed weight along the span is very small compared to the concentrated weights that it may be ignored in frequency calculations. Young's modulus of elasticity (E) = 30×10^6 psi, Moment of inertia (I) = 300 inch^4 . Compute the fundamental natural frequency of the cantilever beam.



Solution:

STEP 1: Remove all weights except the 2000 pound weight at 50" from the fixed end. Since there is no distributed weight, fundamental frequency of this cantilever is the same as a 50" span cantilever with a 2000 pound weight at free end.



For this 50" cantilever, the stiffness k is given in Table 1.

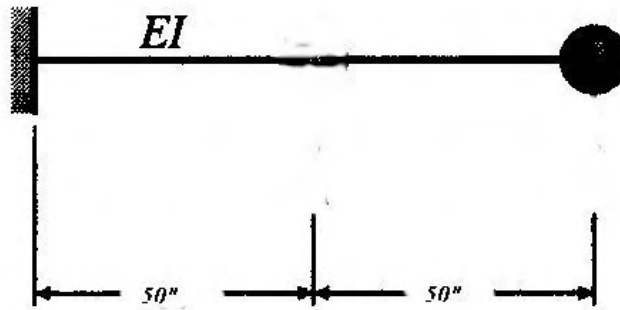
$$k = \frac{3EI}{L^3} = \frac{3(30 \times 10^6) \times 300}{50^3} = 216000$$

$$M = W/g = 2000/386.4 = 5.176 \text{ lb-sec}^2/\text{inch}$$

Using the notations in Section 6,

$$\Omega_1^2 = \frac{k}{M} = \frac{216000}{5.167} = 41804$$

STEP 2: Remove all weights except the 2000 pound weight at the free end.



For this 100" cantilever, the stiffness k is given in Table 1.

$$k = \frac{3EI}{L^3} = \frac{3(30 \times 10^6) \times 300}{100^3} = 216000 = 27000$$

As in STEP 1,

$$M = W/g = 2000/386.4 = 5.176 \text{ lb-sec}^2/\text{inch}$$

So,

$$\Omega_2^2 = \frac{k}{M} = \frac{27000}{5.167} = 5225$$

STEP 3: Using eqn. (6.1), the fundamental natural frequency ω is given by

$$\frac{1}{\omega^2} = \frac{1}{\Omega_1^2} + \frac{1}{\Omega_2^2} = \frac{1}{41804} + \frac{1}{5225} = 0.000215$$

$$\omega^2 = 4651$$

$$\omega = 68.2 \text{ rad/sec}$$

7. INFLUENCE OF MASS AND STIFFNESS

Examining equations 2.1, 3.1 and 4.1, we conclude that:

1. Adding mass to a beam always decreases the fundamental frequency. So, if we have a beam and we place a machine somewhere on the span, fundamental frequency would decrease (if the stiffness remains the same).
2. Adding stiffness to a beam always increases the fundamental frequency (if the mass remains the same). If we increase the cross sectional area of a beam, it not only increases the beam stiffness but also the self-mass of the beam. Whether the fundamental frequency increases, decreases or remains the same depends on the relative increases in mass and stiffness.

8. EFFECT OF ADDITIONAL SUPPORTS

8.1 Rigid Supports

Adding a rigid support (either translational or rotational) increases the stiffness and thus increases the fundamental natural frequency of the beam. For example, compare the fundamental frequency of a clamped-free beam (cantilever) to that of a clamped-simply supported beam (propped cantilever) in Sections 3 and 4. Similarly compare the fundamental frequency of a clamped-simply supported beam (propped cantilever) to that of a clamped-clamped beam. If you add an additional rigid support somewhere in the span to any beam, the effect is to increase the fundamental natural frequency.

8.2 Spring Supports

Adding a spring support (either translational or rotational) also increases the fundamental natural frequency of the beam. Higher the spring stiffness, larger the increase in the fundamental frequency. In the limiting case of infinite stiffness, the spring support becomes a rigid support and provides the maximum increase in the frequency.

Consider a beam with both ends simply supported and let its fundamental natural frequency be ω_{s-s} . Add rotational springs of finite stiffness R at both ends and let the fundamental frequency be ω_{R-R} . Let the fundamental frequency of the corresponding clamped-clamped beam be ω_{c-c} . Based on the preceding discussion,

$$\omega_{s-s} < \omega_{R-R} < \omega_{c-c}$$

Thus we have an upper and a lower bound for the fundamental frequency of a simply supported beam with rotational restraints at ends. Such upper and lower bound values are useful when the spring stiffness is not known or the engineer wants only a rough estimate of the fundamental frequency without having to do a more detailed computation such as finite element analysis.

9. EFFECT OF AXIAL LOAD

Consider a single-span beam of uniform mass and subjected to an axial load P (positive when compressive and negative when tensile). An approximate solution for the fundamental natural frequency of the beam (ω) is

$$\omega^2 = \omega_0^2 \left[1 - \frac{P}{P_c} \right] \quad (9.1)$$

where

ω_0 = Fundamental natural frequency of the beam without the axial load, and

P_c = critical load (buckling load) of the beam.

Critical load of a beam subjected to axial compression (and no other load) is given by

$$P_c = B \frac{EI}{L^2} \quad (9.2)$$

where E , I and L are as defined before and B is the buckling factor listed in Table 3.

Boundary Conditions	Buckling Factor (B)
Both ends simply supported	π^2
Both ends clamped	$4 \pi^2$
One end clamped, other end free (cantilever)	$0.25 \pi^2$
One end clamped, other end simply supported (propped cantilever)	$2.048 \pi^2$

Table 3: Buckling Factor

Beam becomes unstable at $P = P_c$ and the equation is not valid for $P > P_c$. This is a lower bound to the exact result for $0 < P < P_c$; that is, it is a lower bound for compressive loads up to the critical load. According to Lurie [2], eqn. (9.1) is very close to the exact solution for $0 < P < P_c$. Error for tensile loads increases with the magnitude

of the tensile load and could become substantial. From eqn. (9.1), we may note that the effect of compressive load is to decrease the fundamental frequency and the effect of tensile load is to increase it.

Numerical Example-9

Problem:

Consider the beam described in Numerical Example 4. Compute its fundamental natural frequency when it is subjected to a compressive axial load of 50,000 pounds.

Solution:

STEP 1: Compute the fundamental frequency when there is no axial load. This was already done in Numerical Example 4.

$$\omega_0^2 = 28190$$

$$\omega_0 = 168 \text{ rad/sec}$$

In Numerical Example 4, the notation ω_1 was used to denote the fundamental frequency. Here we use ω_0 to be consistent with eqn. (9.1).

STEP 2: Compute the critical load of the beam.

From eqn. (9.2),

$$P_c = B \frac{EI}{L^2}$$

where $B = \pi^2$ (from Table 3 for beams simply supported at both ends)

$$E = 30 \times 10^6 \text{ psi}, \quad I = 400 \text{ inch}^4, \quad L = 200 \text{ inch}$$

Substitution into eqn. (9.2) yields,

$$P_c = 2960000 \text{ lb}$$

STEP 3: Compute the fundamental frequency when subjected to an axial compression of 50000 pounds.

From eqn. (9.1),

$$\omega^2 = \omega_0^2 \left[1 - \frac{P}{P_c} \right]$$

In this problem

$$\omega_0^2 = 28190 \text{ (from Step 1)}$$

$$P_c = 2960000 \text{ lb (from Step 2)}$$

$$P = 50000 \text{ lb (applied axial load)}$$

Substitution of these values into eqn. (9.1) yields,

$$\omega^2 = 23429$$

$$\omega = 153 \text{ rad/sec}$$

This is a lower bound to the exact solution.

REFERENCES

1. Wachel, J. C., and Bates, C. L., "Techniques for controlling piping vibration failures", ASME Paper, 76-Pet-18, 1976.

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